

Differential gorms, differential worms

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Abstract

We study “higher-dimensional” generalizations of differential forms. Just as differential forms can be defined as the universal commutative differential algebra containing $C^\infty(M)$, we can define differential gorms as the universal commutative bidifferential algebra containing $C^\infty(M)$. From a more conceptual point of view, differential forms are functions on the superspace of maps $\mathbb{R}^{0|1} \rightarrow M$ and the action of $\text{Diff}(\mathbb{R}^{0|1})$ on forms is equivalent to deRham differential and to degrees of forms. Gorms are functions on the superspace of maps $\mathbb{R}^{0|2} \rightarrow M$ and we study the action of $\text{Diff}(\mathbb{R}^{0|2})$ on gorms; it contains more than just degrees and differentials. By replacing 2 with arbitrary n , we get differential worms.

We also study a generalization of homological algebra that one obtains by replacing $\text{Diff}(\mathbb{R}^{0|1})$ with $\text{Diff}(\mathbb{R}^{0|n})$ for $n \geq 2$, and the closely related question of forms (and gorms and worms) on some generalized spaces (contravariant functors and stacks) and of approximations of such “spaces” in terms of worms.

Clearly, this is not a gormless paper.

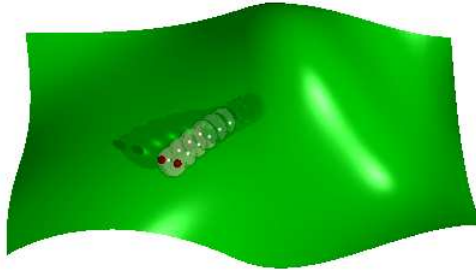


Figure 1: A worm on a manifold

Contents

1	Introduction	2
2	Differential gorms as a universal bidifferential algebra	3
3	Appetizer: differential forms	4
3.1	ΠTM as the space of odd curves	4
3.2	De Rham differential and action of $\text{Diff}(\mathbb{R}^{0 1})$	5
3.3	Cartan formula and its generalizations	5

3.4	Vector fields on ΠTM	6
3.5	Integration and Stokes formula	7
4	Differential gorms as functions on the space of odd surfaces	7
4.1	More than differentials: action of $Diff(\mathbb{R}^{0 2})$	8
4.2	Cartan gormulas	9
4.3	Cohomology of gorms	9
4.4	Integration of gorms	9
5	Differential gorms as a representation of $Diff(\mathbb{R}^{0 2})$	9
5.1	Irreducible representations of $Mat(n)$ and of the categories $Vect$ and $Diff^{op}$. .	10
5.2	Decomposition of $\Omega_{[2]}(M)$	10
5.2.1	Decomposition to $Mat(2)$ irreducibles	10
5.2.2	Generic part: the bundles $\widetilde{T}_l^* M$ (cotangent tetris)	11
5.2.3	An example: $\widetilde{T}_{\boxplus}^* M$	12
5.2.4	Decomposition of the generic part	12
5.2.5	The non-generic part and differential forms	12
5.2.6	The entire decomposition	13
5.3	Derivations on gorms as a module of the crossed product of $C^\infty(\mathbb{R}^{0 2})$ with $Diff(\mathbb{R}^{0 2})$	13
6	Integration and Euler characteristic	14
7	Beyond homological algebra	14
7.1	Example: $n = 1$ (the case of homological algebra)	15
7.2	Representability of functors and their approximations	16
7.3	Examples of approximations	17
7.4	Approximations and representability of stacks	18
7.4.1	Groupoids over a category and Hilsum-Skandalis morphisms	19
7.4.2	The 2-category of Lie groupoids and the stacks they represent	20
7.4.3	Generalized Lie groupoids and the stacks they represent	20
7.5	Examples of stacks and of generalized Lie groupoids	21
7.5.1	“Categorified de Rham complex”	21
7.5.2	Weil and Cartan models of equivariant cohomology	22
7.5.3	Quasi-Poisson groupoids	22

1 Introduction

There is a well known idea of regarding differential forms on a manifold M as functions on a supermanifold, namely on the odd tangent bundle ΠTM . The de Rham differential then becomes a vector field on ΠTM .

This paper is based on the following remarkable fact: one can describe the supermanifold ΠTM as the superspace of all maps $\mathbb{R}^{0|1} \rightarrow M$ (i.e. as the superspace of all odd curves in M). More importantly, the action of the group $Diff(\mathbb{R}^{0|1})$ then gives rise to de Rham differential and to degrees of differential forms. This point of view can be found (more or less explicitly) at many places in physics literature, but we took it explicitly from [Kon].

We use this idea for two purposes. Firstly, it is natural to make a generalization and to study the superspaces of all maps $\mathbb{R}^{0|n} \rightarrow M$ for arbitrary n , and the action of $Diff(\mathbb{R}^{0|n})$

on these superspaces. To explain the title, the functions on these map spaces will be called *differential gorms* in the case of $n = 2$, or generally *differential worms* for arbitrary n . Secondly, the idea is straightforwardly applied to some generalizations of manifolds, namely to contravariant functors from the category of manifolds, or more generally to stacks. A contravariant functor F from the category of manifolds is usually understood as a “generalized space”, such that $F(M)$ is the set of maps from M to that space. Differential forms on F should thus be functions on $F(\mathbb{R}^{0|1})$ (and differential gorms functions on $F(\mathbb{R}^{0|2})$, etc.). As an example, we get a very simple interpretation of equivariant de Rham theory. The problem of worms of generalized spaces is closely related with a generalization of homological algebra, where $Diff(\mathbb{R}^{0|1})$ is replaced by $Diff(\mathbb{R}^{0|n})$ for arbitrary n .

Here is the plan of the paper: In Section 2 we shall describe differential gorms without use of supermanifolds, as the universal commutative bidifferential algebra containing the algebra $C^\infty(M)$. This point of view is not completely satisfactory, as it doesn’t reveal the action of the supergroup $Diff(\mathbb{R}^{0|2})$ (only the sub-supergroup of affine transformations can be seen). Then, as an appetiser, in Section 3 we identify differential forms with functions on the space of all odd curves and derive the basic properties of differential forms from this fact. In Section 4 we start to investigate differential gorms (and worms) as functions on the superspace of all maps $\mathbb{R}^{0|2} \rightarrow M$. In Section 5 we decompose gorms as a representation of $Diff(\mathbb{R}^{0|2})$ and in Section 6 we prove a theorem connecting the Euler characteristic with the integral of any closed integrable gorm. In the final, and possibly the most interesting Section 7, we look at generalized spaces - contravariant functors and stacks.

Finally we should add that the ideas used here are very simple; much of the length of the paper is caused by our attempt to write explicit coordinate expressions.

Remarks on notation

Generally we denote even coordinates on supermanifolds by latin letters and odd coordinates by greek letters. To avoid confusion with differential forms, we denote Berezin integral with respect to (say) x and ξ as $\int f(x, \xi) \overline{dx d\xi}$.

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2 Differential gorms as a universal bidifferential algebra

One can describe the algebra of differential forms $\Omega(M)$ as the universal graded-commutative differential graded algebra containing the algebra $C^\infty(M)$. That is, there is an algebra homomorphism $C^\infty(M) \rightarrow \Omega^0(M)$, and if \mathcal{A} is any graded-commutative differential graded algebra with a homomorphism $C^\infty(M) \rightarrow \mathcal{A}^0$, there is a unique homomorphism of differential graded algebras $\Omega(M) \rightarrow \mathcal{A}$ making the triangle commutative.

In the same spirit, we can define the algebra of *differential gorms* $\Omega_{[2]}(M)$ as the universal graded-commutative bidifferential algebra containing $C^\infty(M)$ (by a bidifferential algebra we mean a bicomplex with a compatible structure of algebra; in particular, it is \mathbb{Z}^2 -graded).

That is, there is an algebra homomorphism $C^\infty(M) \rightarrow \Omega_{[2]}^{0,0}(M)$, and if \mathcal{A} is any graded-commutative bidifferential algebra with a homomorphism $C^\infty(M) \rightarrow \mathcal{A}^{0,0}$, there is a unique homomorphism of bidifferential algebras $\Omega_{[2]}(M) \rightarrow \mathcal{A}$ making the triangle commutative. Just as in the case of $\Omega(M)$ it turns out that $\Omega_{[2]}^{0,0}(M) \cong C^\infty(M)$.

To make this abstract definition down-to-earth, let us choose local coordinates x^i on M . The algebra $\Omega_{[2]}(M)$ is freely generated by the algebra of functions, and by the elements d_1x^i , d_2x^i and $d_1d_2x^i$ (of bidegrees $(1,0)$, $(0,1)$ and $(1,1)$ respectively), where d_a ($a = 1, 2$) are the two differentials in $\Omega_{[2]}(M)$. If f is a function then $d_af = \frac{\partial f}{\partial x^i}d_ax^i$ and therefore $d_1d_2f = \frac{\partial^2 f}{\partial x^i \partial x^j}d_1x^i d_2x^j + \frac{\partial f}{\partial x^i}d_1d_2x^i$. If \tilde{x}^i is another system of local coordinates, we get from here

$$d_a \tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} d_a x^j$$

$$d_1 d_2 \tilde{x}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} d_1 x^j d_2 x^k + \frac{\partial \tilde{x}^i}{\partial x^j} d_1 d_2 x^j.$$

Differential forms are not tensor fields, since their transformation law involves second derivatives; they belong to 2nd order geometry.

Finally, for arbitrary n we can define the algebra of *differential worms of level n* , $\Omega_{[n]}(M)$, as the universal graded-commutative n -differential algebra containing $C^\infty(M)$. One can easily compute transformation laws for worms of any level; they contain n -th derivatives at most.

On $\Omega_{[n]}(M)$ we have an obvious action of the semigroup $Mat(n)$: it leaves $C^\infty(M)$ intact, and linearly transforms the n differentials d_a . It turns out that the differentials and the action of $Mat(n)$ are just the tip of an iceberg: on $\Omega_{[n]}(M)$ we have an action of the supersemigroup of all maps $\mathbb{R}^{0|n} \rightarrow \mathbb{R}^{0|n}$ ($Mat(n)$ corresponds to linear transformations of $\mathbb{R}^{0|n}$ and the differentials to translations). To see this will require a new point of view on differential worms.

3 Appetizer: differential forms

3.1 ΠTM as the space of odd curves

We denote by ΠTM the supermanifold $M^{\mathbb{R}^{0|1}}$ of all maps $\mathbb{R}^{0|1} \rightarrow M$. It is characterized by the following property: for any supermanifold Y , a map $Y \rightarrow \Pi TM$ is the same as a map $\mathbb{R}^{0|1} \times Y \rightarrow M$ (in other words, the functor ΠT is the right adjoint of the functor $Y \mapsto \mathbb{R}^{0|1} \times Y$).

It is easy to understand ΠTM in local coordinates. If θ is the coordinate on $\mathbb{R}^{0|1}$ and x^i are local coordinates on M , a map $\mathbb{R}^{0|1} \rightarrow M$ parametrized by Y , i.e. a map $\mathbb{R}^{0|1} \times Y \rightarrow M$, is given by functions

$$x^i(\theta, \eta) = x^i(\eta) + \theta \xi^i(\eta),$$

where η denotes local coordinates on Y (we just used Taylor expansion in θ). Such a map is therefore the same as a map from Y to a supermanifold with even coordinates x^i and odd coordinates ξ^i . To find ΠTM globally, suppose \tilde{x}^i is another system of local coordinates on M ; then

$$\tilde{x}^i(\theta) = \tilde{x}^i(x(\theta)) = \tilde{x}^i(x + \theta \xi) = \tilde{x}^i(x) + \theta \frac{\partial \tilde{x}^i}{\partial x^j} \xi^j,$$

i.e. the transition functions on ΠTM are

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{\xi}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \xi^j.$$

Now we can see that we can identify functions on ΠTM with differential forms on M , by identifying ξ^i with dx^i (we also see that our notation is correct, i.e. that the supermanifold is indeed the odd tangent bundle of M).

3.2 De Rham differential and action of $Diff(\mathbb{R}^{0|1})$

Since ΠTM is the supermanifold of all maps $\mathbb{R}^{0|1} \rightarrow M$, we have an action of the supergroup $Diff(\mathbb{R}^{0|1})$ on it. Let us first describe $Diff(\mathbb{R}^{0|1})$. It is an open sub-supergroup of the super-semigroup $\Pi T\mathbb{R}^{0|1}$ of all maps $\mathbb{R}^{0|1} \rightarrow \mathbb{R}^{0|1}$. A (possibly parametrized) map $\mathbb{R}^{0|1} \rightarrow \mathbb{R}^{0|1}$ is of the form $\theta' = a\theta + \beta$, hence the supersemigroup is diffeomorphic to $\mathbb{R}^{1|1}$, with one even coordinate a and one odd coordinate β ; $Diff(\mathbb{R}^{0|1})$ is given by $a \neq 0$.

The right action of $\Pi T\mathbb{R}^{0|1}$ (hence also of $Diff(\mathbb{R}^{0|1})$) on ΠTM is given by $x' + \theta\xi' = x + (a\theta + \beta)\xi$, i.e.

$$x' = x + \beta\xi, \quad \xi' = a\xi.$$

The vector fields generating this action are

$$E = \xi^i \frac{\partial}{\partial \xi^i}, \quad d = \xi^i \frac{\partial}{\partial x^i}.$$

When E acts on a function on ΠTM , i.e. on a differential form on M , it multiplies it by its degree; on the other hand, d acts as the de Rham differential. The canonical structure of a complex on $\Omega(M)$ is therefore equivalent to the action of $Diff(\mathbb{R}^{0|1})$. The fact that it is a complex follows from the commutation relations

$$[E, d] = d, \quad [d, d] = 0$$

in the Lie algebra of $Diff(\mathbb{R}^{0|1})$.

3.3 Cartan formula and its generalizations

Now we will try to understand Cartan's formula $\mathcal{L}_v = di_v + i_v d$ from this point of view. It will be convenient to treat a general map space Y^X of all maps $X \rightarrow Y$ between two (super)manifolds; we shall however pretend that Y^X is finite-dimensional (since it is so for $X = \mathbb{R}^{0|n}$) to avoid analytical problems. As we shall see, Cartan's formula comes from the action of the group $Diff(X) \ltimes (Diff(Y))^X$ on Y^X (in the case of $X = \mathbb{R}^{0|1}$).

Let $ev : X \times Y^X \rightarrow Y$ be the evaluation map (adjoint to the identity map $Y^X \rightarrow Y^X$); in the case of $X = \mathbb{R}^{0|1}$ and $Y = M$ (so that $Y^X = \Pi TM$), $ev : (\theta, x, \xi) \mapsto x + \theta\xi$. We can naturally identify vector fields on Y^X with sections of the vector bundle ev^*TY ,¹ and therefore we can multiply them with arbitrary functions on $X \times Y^X$ (not just on Y^X). If f is a function on X (and therefore also on $X \times Y^X$) and w a vector field on Y^X , we shall denote their product as $f \cdot w$.

If u is a vector field on X and v a vector field on Y , we shall denote their natural lifts to Y^X as u^\flat and v^\sharp ; these are the vector fields generating the *left* actions of $Diff(X)$ and $Diff(Y)$ on Y^X . The following formulas express the fact that on Y^X we have a (left) action of the semidirect product $Diff(X) \ltimes (Diff(Y))^X$:

$$[u_1^\flat, u_2^\flat] = [u_1, u_2]^\flat$$

¹to see this, realize that a vector field on Y^X is an infinitesimal deformation of the identity map $Y^X \rightarrow Y^X$, i.e. an infinitesimal deformation of $ev : X \times Y^X \rightarrow Y$, i.e a section of ev^*TY

$$[u^\flat, f \cdot v^\sharp] = (uf) \cdot v^\sharp$$

$$[f_1 \cdot v_1^\sharp, f_2 \cdot v_2^\sharp] = (-1)^{|f_2||v_1|} f_1 f_2 \cdot [v_1, v_2]^\sharp;$$

we also have

$$f \cdot u^\flat = (fu)^\flat.$$

In the case of $X = \mathbb{R}^{0|1}$ and $Y = M$ we have $(\partial_\theta)^\flat = -d$, $(\theta\partial_\theta)^\flat = -E$ (the minus signs are here since d and E generate the *right* action), $\theta \cdot \partial_{x^i} = -\partial_{\xi^i}$ (and therefore $\theta \cdot \partial_{\xi^i} = 0$). Moreover, for any vector field v on M we have $v^\sharp = \mathcal{L}_v$ and $\theta \cdot v^\sharp = -i_v$. The equation $[(\partial_\theta)^\flat, \theta \cdot v^\sharp] = v^\sharp$ is Cartan's $d i_v + i_v d = \mathcal{L}_v$.

3.4 Vector fields on ΠTM

As we have seen, the space of vector fields on Y^X is a module over $C^\infty(X)$, and it is also a representation of $Diff(X)$. If $\phi \in Diff(X)$, $f \in C^\infty(X)$ and w is a vector field on Y^X then clearly

$$\phi \cdot (f \cdot w) = (f \circ \phi^{-1}) \cdot (\phi \cdot w);$$

in other words, the space of vector fields on Y^X is a module over the crossed product of $C^\infty(X)$ with $Diff(X)$. Infinitesimally, if u is a vector field on X ,

$$[u^\flat, f \cdot w] = (uf) \cdot w + (-1)^{|f||u|} f \cdot [u^\flat, w].$$

Now let us return to the case of $X = \mathbb{R}^{0|1}$, $Y = M$, $Y^X = \Pi TM$. Vector fields on Y^X are then derivations of the algebra $\Omega(M)$. Notice that $\theta \cdot (\theta \cdot w) = 0$ (since $\theta^2 = 0$) and that $[(\partial_\theta)^\flat, [(\partial_\theta)^\flat, w]] = 0$ (since $[(\partial_\theta)^\flat, (\partial_\theta)^\flat] = 0$), i.e. both θ and $(\partial_\theta)^\flat$ act as differentials; $(\partial_\theta)^\flat$ increases degree by 1, while θ decreases it by 1. Finally,

$$w = [(\partial_\theta)^\flat, \theta \cdot w] + \theta \cdot [(\partial_\theta)^\flat, w],$$

i.e. any w can be *uniquely* decomposed as $w = w_1 + w_2$ (by $w_1 = [(\partial_\theta)^\flat, \theta \cdot w]$, $w_2 = \theta \cdot [(\partial_\theta)^\flat, w]$) so that $[(\partial_\theta)^\flat, w_1] = 0$ and $\theta \cdot w_2 = 0$.

In coordinates, a vector field w on ΠTM , of degree p and such that $\theta \cdot w = 0$, is of the form

$$w = A_{i_1 i_2 \dots i_{p+1}}^k \xi^{i_1} \xi^{i_2} \dots \xi^{i_{p+1}} \partial_{\xi^k},$$

i.e. it is a section of $TM \otimes \bigwedge^{p+1} T^*M$.

Let us summarize what we have found. The graded Lie algebra $Der(\Omega(M))$ decomposes to a direct sum of graded vector spaces

$$Der(\Omega(M)) = \ker(\theta) \oplus \ker(\partial_\theta)$$

(they turn out to be subalgebras). The two subspaces are naturally isomorphic; the two mutually inverse isomorphisms are the action of θ ($\ker(\partial_\theta) \rightarrow \ker(\theta)$) and the action of ∂_θ ($\ker(\theta) \rightarrow \ker(\partial_\theta)$). Moreover, they are both isomorphic with the space of vector fields with values in differential forms. The Lie bracket on $\ker(\partial_\theta)$ (the derivations of $\Omega(M)$ commuting with the differential) is the Frölicher-Nijenhuis bracket.

3.5 Integration and Stokes formula

The fact that ΠTM is a map space doesn't seem to shed much light on integration of differential forms, i.e. of functions on ΠTM . For this reason we just repeat the simple standard facts: the volume measure $\overline{dx d\xi}$ on ΠTM is independent of the choice of coordinate system x^i (up to choice of orientation), and it is also invariant with respect to the vector field d . For that reason $\int d\alpha = 0$ for any form α with compact support. To get Stokes theorem, let χ_Ω be the characteristic function of a compact domain Ω ; then $0 = \int d(\chi_\Omega \alpha) = \int (d\chi_\Omega) \alpha + \int \chi_\Omega d\alpha = -\int_{\partial\Omega} \alpha + \int_\Omega d\alpha$.

It is certainly an interesting thing that on the space of all maps $\mathbb{R}^{0|1} \rightarrow M$ there is a natural volume measure (e.g. from the point of view of quantum field theory). As we will see, the situation gets even better for $\mathbb{R}^{0|n}$ with $n \geq 2$; the measure will be $Diff(\mathbb{R}^{0|n})$ -invariant (here it was d -invariant, but not E -invariant).

4 Differential gorms as functions on the space of odd surfaces

Everything we'll be doing here will be fairly analogous to the previous section, so we can be brief. Let $(\Pi T)^2 M$ denote the supermanifold of all maps $\mathbb{R}^{0|2} \rightarrow M$. Let θ^1, θ^2 be the coordinates on $\mathbb{R}^{0|2}$ and x^i be local coordinates on M . A (parametrized) map $\mathbb{R}^{0|2} \rightarrow M$ expanded to Taylor series in θ 's looks as

$$x^i(\theta^1, \theta^2) = x^i + \theta^1 \xi_1^i + \theta^2 \xi_2^i + \theta^2 \theta^1 y^i,$$

$(\Pi T)^2 M$ has therefore local coordinates x^i, y^i (even coordinates) and ξ_1^i, ξ_2^i (odd coordinates). If \tilde{x}^i is another system of local coordinates on M then (expanding to Taylor series)

$$\begin{aligned} \tilde{x}^i(x(\theta^1, \theta^2)) &= \tilde{x}^i(x + \theta^1 \xi_1 + \theta^2 \xi_2 + \theta^2 \theta^1 y) = \\ &= \tilde{x}^i(x) + \theta^1 \frac{\partial \tilde{x}^i}{\partial x^j} \xi_1^j + \theta^2 \frac{\partial \tilde{x}^i}{\partial x^j} \xi_2^j + \theta^2 \theta^1 \left(\frac{\partial \tilde{x}^i}{\partial x^j} y^j + \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \xi_1^j \xi_2^k \right), \end{aligned}$$

i.e. the transition functions on $(\Pi T)^2 M$ are

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{\xi}_a^i = \frac{\partial \tilde{x}^i}{\partial x^j} \xi_a^j, \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j + \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \xi_1^j \xi_2^k.$$

From this we can see that we can identify differential gorms on M (as defined in section 2) with functions on $(\Pi T)^2 M$ polynomial in y 's, by identifying $d_a x^i$ with ξ_a^i and $d_1 d_2 x^i$ with y^i . We arrived to this identification by a computation, but there is a simpler reason using differentials (see below). General functions on $(\Pi T)^2 M$ (not necessarily polynomial in y 's) will be called *pseudodifferential gorms* (these are things like $e^{-(d_1 d_2 x)^2} d_1 x d_2 x$).

Let us notice that the body of the supermanifold $(\Pi T)^2 M$ is naturally isomorphic to TM ; indeed, we get the body by setting all the odd coordinates ξ_a^i to zero, and then the transition functions for x 's and y 's become those of TM .

Finally, let us describe $(\Pi T)^n M = M^{\mathbb{R}^{0|n}}$ for higher n 's and directly identify its functions with differential worms of level n on M . If x^i 's are local coordinates on M then the coordinates on $(\Pi T)^n M$ are $x^i, d_a x^i$ ($1 \leq a \leq n$), $d_a d_b x^i$ ($1 \leq a < b \leq n$), \dots , $d_1 d_2 \dots d_n x^i$ (i.e. apply the differentials d_a ($1 \leq a \leq n$) to x^i 's in all possible ways). These coordinates are identified with Taylor coefficients of a map $\mathbb{R}^{0|n} \rightarrow M$ by

$$x^i(\theta^1, \theta^2, \dots, \theta^n) = e^{\theta^a d_a x^i}.$$

From this expression it is clear that $-(\partial_{\theta^a})^b$ is equal to d_a . For example, in the case of $n = 2$ we have $-(\partial_{\theta^1})^b = \xi_1^i \partial_{x^i} + y^i \partial_{\xi_2^i}$, $-(\partial_{\theta^2})^b = \xi_2^i \partial_{x^i} - y^i \partial_{\xi_1^i}$, i.e. $-(\partial_{\theta^a})^b = \xi_a^i \partial_{x^i} + \epsilon_{ab} y^i \partial_{\xi_b^i}$.

4.1 More than differentials: action of $\text{Diff}(\mathbb{R}^{0|2})$

We have already found the vector fields $-(\partial_{\theta^a})^b$: they generate the action of the group of translations of $\mathbb{R}^{0|2}$ on $(\Pi T)^2 M$, and they are equal to the two differentials on differential gorms. We can easily find u^b for any vector field u on $\mathbb{R}^{0|2}$: either we do it directly, regarding u as an infinitesimal transformation on $\mathbb{R}^{0|2}$ and finding the corresponding infinitesimal transformation of $(\Pi T)^2 M$ from the formula

$$x' + \theta^a \xi'_a + \theta^2 \theta^1 y' = x + \theta'^a \xi_a + \theta'^2 \theta'^1 y, \quad (1)$$

or use the known expression for $(\partial_{\theta^a})^b$ and the identities

$$\theta^a \cdot \partial_{x^i} = -\partial_{\xi_a^i}, \quad \theta^2 \theta^1 \cdot \partial_{x^i} = \partial_{y^i}.$$

The result is

$$\begin{aligned} d_a &= -(\partial_{\theta^a})^b = \xi_a^i \partial_{x^i} + \epsilon_{ab} y^i \partial_{\xi_b^i} \\ E_a^b &= -(\theta^b \partial_{\theta^a})^b = \xi_a^i \partial_{\xi_b^i} + \delta_b^a y^i \partial_{y^i} \\ R_a &= -(\theta^2 \theta^1 \partial_{\theta^a})^b = \xi_a^i \partial_{y^i}. \end{aligned}$$

These vector fields give us a right action of the Lie algebra $\mathfrak{diff}(\mathbb{R}^{0|2})$ on $(\Pi T)^2 M$ (that is the reason for the minus signs: u^b give the left action), i.e. a left action on the algebra of differential gorms. E_1^1 and E_2^2 are the two degrees on differential gorms, E_a^b generate the action of $\mathfrak{gl}(2)$ on gorms, but R_a 's are something new. Similar formulas can be easily found for worms of arbitrary level.

If we want to know the right action of the supersemigroup $(\Pi T)^2 \mathbb{R}^{0|2} = (\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$ on $(\Pi T)^2 M$ (not just the infinitesimal action we have just derived), we can easily find it from (1). A (parametrized) map $\mathbb{R}^{0|2} \rightarrow \mathbb{R}^{0|2}$ is of the form

$$\begin{aligned} \theta'^1 &= \beta^1 + a_1^1 \theta^1 + a_2^1 \theta^2 + \gamma^1 \theta^2 \theta^1 \\ \theta'^2 &= \beta^2 + a_1^2 \theta^1 + a_2^2 \theta^2 + \gamma^2 \theta^2 \theta^1, \end{aligned}$$

i.e. $(\Pi T)^2 \mathbb{R}^{0|2}$ is diffeomorphic to $\mathbb{R}^{4|4}$ (with even coordinates a_b^a and odd coordinates β^a, γ^a); we also see that its body is the semigroup $\text{Mat}(2)$ of 2×2 -matrices. The action of $\text{Mat}(2)$ (i.e. when we set β 's and γ 's to zero) is given by

$$x' = x, \quad \xi'_a = a_a^b \xi_b, \quad y' = \det(A) y,$$

where A is the matrix a_b^a . If it ever becomes useful, the full result of (1) is

$$x' = x + \beta^a \xi_a + \beta^2 \beta^1 y, \quad \xi'_a = a_a^b \xi_b + \epsilon_{bc} \beta^b a_a^c y, \quad y' = (\det(A) + \epsilon_{bc} \beta^b \gamma^c) y + \gamma^a \xi_a.$$

4.2 Cartan gormulas

Analogues of the Cartan formula $\mathcal{L}_v = d i_v + i_v d$ (i.e. $v^\sharp = [(\partial_\theta)^\flat, \theta \cdot v^\sharp]$) for worms of arbitrary level were already found in section 3.3. Here we mention just one special case: since $\theta^2 \theta^1 \cdot \partial_{x^i} = \partial_{y^i}$, we have $\theta^2 \theta^1 \cdot v^\sharp = v^i \partial_{y^i}$ for any vector field $v = v^i \partial_{x^i}$ on M ; let us denote $\theta^2 \theta^1 \cdot v^\sharp$ as i_v . Then $[d_1, [d_2, i_v]] = \mathcal{L}_v$.

4.3 Cohomology of gorms

The cohomology of differential gorms with respect to any d_a is naturally isomorphic to de Rham cohomology of M . To see this we just write $(\Pi T)^2 M$ as $\Pi T(\Pi T M)$, i.e. the cohomology of differential gorms is de Rham cohomology of $\Pi T M$, and $\Pi T M$ can be contracted to M .

We can also compute cohomology with respect to say R_1 , since $(R_1)^2 = 0$. This cohomology is isomorphic to $\Omega(M)$. In fact, the projection $\Omega_{[2]}(M) \rightarrow \Omega(M)$ given by $d_1 \mapsto d$, $d_2 \mapsto 0$, is a quasiisomorphism, when $\Omega(M)$ is taken with zero differential and $\Omega_{[2]}(M)$ with differential R_1 . The reason is simple: $[R_1, d_2] = -E_1^1$, i.e. E_1^1 acts trivially on cohomology, i.e. the semigroup action $x \mapsto x$, $d_1 x \mapsto \lambda d_1 x$, $d_2 x \mapsto d_2 x$, $d_1 d_2 x \mapsto \lambda d_1 d_2 x$ is trivial on cohomology, and setting $\lambda = 0$ gives us the result. Notice that we could use the same argument to show that the cohomology with respect to d_2 is just the cohomology of $\Omega(M)$ with respect to d . In geometric terms, we have an embedding $\Pi T M \subset (\Pi T M)^2$ coming from the projection $\mathbb{R}^{0|2} \rightarrow \mathbb{R}^{0|1}$, $(\theta^1, \theta^2) \mapsto \theta^2$; this projection can be obtained by the semigroup action $(\theta^1, \theta^2) \mapsto (\lambda \theta^1, \theta^2)$, setting $\lambda = 0$. The action is generated by $\theta^1 \partial_{\theta^1}$ and $\theta^1 \partial_{\theta^1} = [\theta^2 \theta^1 \partial_{\theta^1}, \partial_{\theta^2}]$.

4.4 Integration of gorms

Like differential forms, gorms (and worms) can be integrated. However we need pseudodifferential gorms (depending non-polynomially on $d_1 d_2 x$'s) for the integral to be finite. In coordinates, the integral is just the Berezin integral with volume measure $\overline{dx^1 d\xi_1^1 d\xi_2^1 dy^1 \dots dx^m d\xi_1^m d\xi_2^m dy^m}$, where m is the dimension of M . In other words, to integrate a gorm, expand it in ξ 's, take the coefficient in front of $\xi_1^1 \xi_2^1 \xi_1^2 \xi_2^2 \dots \xi_1^m \xi_2^m$, and integrate it over y 's and x 's. For example, if $M = \mathbb{R}$,

$$\int e^{-x^2 - (d_1 d_2 x)^2} d_1 x d_2 x = \pi.$$

The integral is independent of the choice of coordinates, i.e. $\overline{dx d\xi_1 d\xi_2 dy}$ is $Diff(M)$ -invariant, and in fact it is also $Diff(\mathbb{R}^{0|2})$ -invariant. It means that for any integrable gorm α and any vector field u on $\mathbb{R}^{0|2}$ we have $\int (u^\flat \alpha) = 0$ – a form of Stokes theorem.

A similar claim is true for worms with any $n \geq 2$; the coordinate Berezin integral on $(\Pi T)^n M$ is both $Diff(\mathbb{R}^{0|n})$ and $Diff(M)$ -invariant.

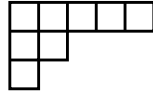
5 Differential gorms as a representation of $Diff(\mathbb{R}^{0|2})$

Our aim in this section is to decompose $\Omega_{[2]}(M)$ to indecomposable representations of $Diff(\mathbb{R}^{0|2})$. It should be compared with the de Rham cohomology of $\Omega(M)$; the latter is connected with the problem "solve the equation $d\alpha = \beta$ in $\Omega(M)$ ", while the decomposition of $\Omega_{[2]}(M)$ describes solutions of all linear equations in $\Omega_{[2]}(M)$ that use the operators d_a , E_a^b and R_a (i.e. the action of $Diff(\mathbb{R}^{0|2})$) on $\Omega_{[2]}(M)$.

The representation theory of $Diff(\mathbb{R}^{0|n})$ for $n \geq 3$ was shown to be wild by N. Shomron [Sho] (i.e. it is impossible to classify all finite-dimensional indecomposable representations of this supergroup). The decomposition of $\Omega_{[n]}$ for $n \geq 3$ remains an open problem for us.

5.1 Irreducible representations of $Mat(n)$ and of the categories $Vect$ and $Diff^{op}$

Let us recall that irreducible representations of the semigroup $Mat(n)$, or more invariantly, of the semigroup $End(V)$, where V is an n -dimensional vector space, are classified by highest weights, that is by n -tuples of integers $l_1 \geq l_2 \geq \dots \geq l_n \geq 0$. Such an n -tuple can be represented by a Young table; for example, the triple $(5, 2, 1)$ is represented by



The representation with the highest weight $\vec{l} = (l_1, l_2, \dots, l_n)$ can be found in the decomposition of the representation $V^{\otimes N}$ to irreducibles, where $N = l_1 + l_2 + \dots + l_n$. Namely, let $W_{\vec{l}}$ be the irreducible representation of the symmetric group S_N , corresponding to the Young table \vec{l} . S_N acts also on $V^{\otimes N}$, by permutations. The irreducible representation of $End(V)$ with the highest weight \vec{l} is then

$$V_{\vec{l}} = Hom_{S_N}(W_{\vec{l}}, V^{\otimes N}).$$

In fact, this formula gives us a representation of the category $Vect$ of finite-dimensional vector spaces, i.e. a functor $Vect \rightarrow Vect$, $V \mapsto V_{\vec{l}}$.

For any manifold M we can consider the space of sections of the vector bundle $T_{\vec{l}}^*M$; it is a right representation of the semigroup of all smooth maps $M \rightarrow M$. This bundle is non-zero iff the number of rows of \vec{l} is at most $\dim M$, and the representation is known to be irreducible unless \vec{l} has only one column; in that case, $T_{\vec{l}}^*M = \bigwedge^N T^*M$, where N is the length of the column, and the space of differential N -forms on M has the invariant subspace of all closed N -forms. Generally, complete reducibility doesn't hold for the representations of these semigroups; we shall meet many examples soon.

Let us notice that the construction $M \mapsto \Gamma(T_{\vec{l}}^*M)$ is a contravariant functor from the category $Diff$ of smooth manifolds to the category of vector spaces. The functor $\Gamma(T_{\vec{l}}^*)$ can also be applied to supermanifolds. Let us however notice that if X is a supermanifold, the action of S_N on $(T^*)^{\otimes N}X$ is modified by the sign rule: the transposition acts by $a \otimes b \mapsto (-1)^{|a||b|}b \otimes a$. This implies that $T_{\vec{l}}^*\mathbb{R}^{0|n}$ is zero iff \vec{l} has more than n columns. For example, there is no Riemann metric on $\mathbb{R}^{0|1}$ (the Young table is $\square\square$), but there are non-zero k -forms for any k .

$\Omega_{[2]}$ is also a right representation of the category $Diff$ (i.e. a contravariant functor from $Diff$), and it is also a representation of the semigroup $(\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$. Our aim will be to decompose it to indecomposable parts.

5.2 Decomposition of $\Omega_{[2]}(M)$

5.2.1 Decomposition to $Mat(2)$ irreducibles

In this preliminary section we decompose $\Omega_{[2]}(M)$ as a representation of $Mat(2) \subset (\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$. It contains only the representations with highest weights (l_1, l_2) such that $l_1 - l_2 \leq \dim M$;

the picture looks something like this (for $\dim M = 4$):

etc...

$$\begin{array}{c}
 5 \\
 4 \\
 3 \\
 l_2 \uparrow 2 \\
 1 \\
 0
 \end{array}
 \begin{array}{cccccccccc}
 & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
 \end{array}
 \begin{array}{c}
 0 \\
 1 \\
 2 \\
 3 \\
 4 \\
 5 \\
 6 \\
 7 \\
 8 \\
 9
 \end{array}
 \begin{array}{c}
 \\
 \\
 \\
 \\
 \\
 l_1 \rightarrow
 \end{array}
 \quad (2)$$

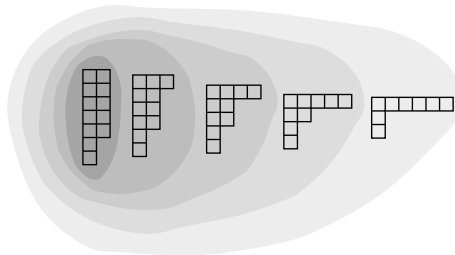
These representations can be found by looking for gorms with highest weights. For example, a general gorm of weight (or bidegree) $(2, 1)$ is of the form $a_{ij}(x)y^i\xi_1^j + b_{ijk}(x)\xi_1^i\xi_1^j\xi_2^k$; the highest weight condition means that it is annulled by $E_1^2 = \xi_1^i\partial_{\xi_1^i}$, in this case it means that $b_{ijk}(x)\xi_1^i\xi_1^j\xi_1^k = 0$, i.e. b_{ijk} becomes 0 after complete skew-symmetrization in ijk .

5.2.2 Generic part: the bundles \widetilde{T}_l^*M (cotangent tetris)

Now we would like to decompose $\Omega_{[2]}(M)$ as a representation of $(\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$ (for a review of the needed representation theory see [Lei]). Let us first describe the *right* irreducible representations of $(\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$; the left irreducible representations are their duals. For any Young table \vec{l} with two columns the representation $\Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})$ is irreducible; these are called *generic* irreducibles. The remaining irreducibles are the spaces of closed differential k -forms on $\mathbb{R}^{0|2}$ for any k . The representation theory of $(\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$ is quite simple: the generic irreducibles can't appear in the composition series of any reducible indecomposable representation.

There is a simple geometrical (and somewhat tautological) way to get an intertwining map $\Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})^* \rightarrow \Omega_{[2]}(M)$, i.e. an equivariant map $(\Pi T)^2M \rightarrow \Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})$, using a section s of $T_{\vec{l}}^*M$: for any map $\phi: \mathbb{R}^{0|2} \rightarrow M$ we have the section ϕ^*s of $T_{\vec{l}}^*\mathbb{R}^{0|2}$, i.e. we have a map from $(\Pi T)^2M = M^{\mathbb{R}^{0|2}}$ to $\Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})$. This is in some sense the central idea of this section.

Let us now try directly to find the space $K_{\vec{l}}(M) = \text{Hom}_{(\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}}(\Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})^*, \Omega_{[2]}(M))$. We have already found a map $\Gamma(T_{\vec{l}}^*M) \rightarrow K_{\vec{l}}(M)$; we shall construct a natural vector bundle $\widetilde{T}_{\vec{l}}^*M$ over M such that $K_{\vec{l}}(M) = \Gamma(\widetilde{T}_{\vec{l}}^*M)$. The bundle $\widetilde{T}_{\vec{l}}^*M$ will come with a natural filtration $F_1 \subset F_2 \subset \dots \subset \widetilde{T}_{\vec{l}}^*M$, such that $F_1 = T_{\vec{l}}^*M$ and the quotients F_i/F_{i-1} are $T_{\vec{k}}^*M$'s for various \vec{k} 's. The rule for getting these \vec{k} 's from \vec{l} should be obvious from this example:



In other words, we start with a two-column table \vec{l} and we keep removing squares from the columns and adding a square to the first row until the second column has length one.

Let us finally find the spaces $K_{\vec{l}}(M)$, describing the bundles $\widetilde{T}_{\vec{l}}^*M$ at the same time. An intertwining map $\Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})^* \rightarrow \Omega_{[2]}(M)$ is the same as a gorm with the weight \vec{l}^T (\vec{l}^T denotes the Young table \vec{l} reflected with respect to the diagonal, i.e. \vec{l}^T is a two-row table), annulled by $E_1^2 = \xi_1^i \partial_{\xi_2^i}$ (i.e. a highest-weight gorm) and by the two operators $R_a = \xi_a^i \partial_{y^i}$. This gorm is the image of the element of $\Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})^*$ that assigns to any section of $T_{\vec{l}}^*\mathbb{R}^{0|2}$ the highest-weight component of its value at the origin of $\mathbb{R}^{0|2}$. The reason for \vec{l}^T is that the action of S_N on $(T^*)^{\otimes N}\mathbb{R}^{0|2}$ is modified by the sign rule, which is equivalent to the reflection of Young tables.

The vector bundle $\widetilde{T}_{\vec{l}}^*M$ is thus the vector bundle of gorms of bidegree \vec{l}^T that are annulled by $\xi_1^i \partial_{\xi_2^i}$ and by R_a 's. The filtration on $\widetilde{T}_{\vec{l}}^*M$ can be described as follows. Let us embed M to $(\Pi T)^2 M$ as the space of constant maps $\mathbb{R}^{0|2} \rightarrow M$; in coordinates, it is given by setting ξ 's and y 's to zero. On $\Omega_{[2]}(M)$ we have the decreasing filtration by the order of vanishing on M (i.e. by the number of y 's and ξ 's). The filtration on $\widetilde{T}_{\vec{l}}^*M$ is the restriction of this filtration.

5.2.3 An example: $\widetilde{T}_{\boxplus}^*M$

Let us compute an example, for the Young table \boxplus . A general gorm with bidegree $(2, 2)$ is of the form $a_{ij}(x)y^i y^j + b_{ijk}(x)y^i \xi_1^j \xi_2^k + c_{ijkl}(x)\xi_1^i \xi_1^j \xi_2^k \xi_2^l$. It is annulled by $E_2^1 = \xi_2^i \partial_{\xi_1^i}$ iff b_{ijk} is symmetric in jk and c_{ijkl} has the symmetries of the Riemann curvature tensor (i.e. it has the symmetries given by the Young table \boxplus). It is annulled by $R_a = \xi_a^i \partial_{y^i}$ iff $a_{ij} = 0$ and b_{ijk} is completely symmetric, i.e. it has the symmetries given by the Young table $\square\square\square$. The pair (b_{ijk}, c_{ijkl}) is a section of $\widetilde{T}_{\boxplus}^*M$. The subbundle $T_{\boxplus}^*M \subset \widetilde{T}_{\boxplus}^*M$ is given by $b_{ijk} = 0$; the quotient $\widetilde{T}_{\boxplus}^*M/T_{\boxplus}^*M$ is clearly $T_{\square\square\square}^*M$.

Similar coordinate computation can be done for arbitrary two-column Young table \vec{l} ; it gives the filtration on $\widetilde{T}_{\vec{l}}^*M$ with the Young tables as drawn on the picture above.

5.2.4 Decomposition of the generic part

We can conclude that the generic part of $\Omega_{[2]}(M)$ is

$$\bigoplus_{\vec{l}} \Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})^* \otimes \Gamma(\widetilde{T}_{\vec{l}}^*M),$$

where we sum over all two-column Young tables \vec{l} . To make this formula more symmetric, let us notice that for any such \vec{l} , $\widetilde{T}_{\vec{l}}^*\mathbb{R}^{0|2} = T_{\vec{l}}^*\mathbb{R}^{0|2}$, since $T_{\vec{k}}^*\mathbb{R}^{0|2} = 0$ for any \vec{k} with at least 3 columns. The generic part of $\Omega_{[2]}(M)$ is thus

$$\bigoplus_{\vec{l}} \Gamma(T_{\vec{l}}^*\mathbb{R}^{0|2})^* \otimes \Gamma(\widetilde{T}_{\vec{l}}^*M).$$

5.2.5 The non-generic part and differential forms

Let us finish the decomposition of $\Omega_{[2]}(M)$ by describing its non-generic part. If $\alpha \in \Omega(M)$ then for any map $\phi : \mathbb{R}^{0|2} \rightarrow M$ we have the differential form $\phi^*\alpha$ on $\mathbb{R}^{0|2}$, i.e. α gives us

an equivariant map $(\Pi T)^2 M \rightarrow \Omega(\mathbb{R}^{0|2})$, i.e. an intertwining map $\Omega(\mathbb{R}^{0|2})^* \rightarrow \Omega_{[2]}(M)$. This map depends on α , i.e. we have found an intertwining map

$$\Omega(\mathbb{R}^{0|2})^* \otimes \Omega(M) \rightarrow \Omega_{[2]}(M). \quad (3)$$

The image of this map is easily seen to be the whole non-generic part of $\Omega_{[2]}(M)$. The map is not injective ($\Omega(\mathbb{R}^{0|2})^*$ is not irreducible); we have to describe its kernel.

The kernel is given by the following simple fact: the map from $\Omega(M)$ to the space of equivariant maps $(\Pi T)^2 M \rightarrow \Omega(\mathbb{R}^{0|2})$ is a morphism of cochain complexes; in other words, the map (3) is also a morphism of cochain complexes, where $\Omega_{[2]}(M)$ is understood as a cochain complex in the trivial way, i.e. it is entirely in degree 0. It means that the elements of $\Omega(\mathbb{R}^{0|2})^* \otimes \Omega(M)$ of non-zero degree are mapped to zero, and the same is the fate of the exact elements of degree 0. One can easily verify that this is the entire kernel, hence the non-generic part of $\Omega_{[2]}(M)$ is isomorphic to the degree-0 part of $\Omega(\mathbb{R}^{0|2})^* \otimes \Omega(M)$ modulo exact elements,

$$\frac{\bigoplus_k \Omega^k(\mathbb{R}^{0|2})^* \otimes \Omega^k(M)}{d(\bigoplus_k \Omega^k(\mathbb{R}^{0|2})^* \otimes \Omega^{k-1}(M))}.$$

5.2.6 The entire decomposition

If we put the generic and the non-generic part of $\Omega_{[2]}(M)$ together, we have the isomorphism

$$\Omega_{[2]}(M) \cong \left(\bigoplus_{\tilde{l}} \Gamma(T_{\tilde{l}}^* \mathbb{R}^{0|2})^* \otimes \Gamma(\widetilde{T_{\tilde{l}}^* M}) \right) \oplus \frac{\bigoplus_k \Omega^k(\mathbb{R}^{0|2})^* \otimes \Omega^k(M)}{d(\bigoplus_k \Omega^k(\mathbb{R}^{0|2})^* \otimes \Omega^{k-1}(M))};$$

this isomorphism is $(\mathbb{R}^{0|2})^{\mathbb{R}^{0|2}}$ -equivariant and functorial in M (in particular, it is $Diff(M)$ -equivariant).

5.3 Derivations on gorms as a module of the crossed product of $C^\infty(\mathbb{R}^{0|2})$ with $Diff(\mathbb{R}^{0|2})$

In this section we briefly apply the general formulas of section 3.4 to vector fields on $(\Pi T)^2 M$. The action of θ_1 , θ_2 , d_1 and d_2 on the space $Der(\Omega_{[2]}(M))$ of these vector fields generate an action of a Clifford algebra. As a result, we have an isomorphism

$$Der(\Omega_{[2]}(M)) \cong Der(\Omega_{[2]}(M))_0 \otimes \bigwedge(\mathbb{R}^2),$$

where $Der(\Omega_{[2]}(M))_0$ is the space of vector fields annulled by θ_1 and θ_2 and \mathbb{R}^2 is the vector space with the basis d_1 and d_2 . A vector field $A^i \partial_{x^i} + B_a^i \partial_{\xi^i} + C^i \partial_{y^i}$ is annulled by both θ 's iff $A^i = B_a^i = 0$. We can thus naturally identify $Der(\Omega_{[2]}(M))_0$ with the space of gorm-valued vector fields on M . Similar result holds for worms of arbitrary level.

To give a decomposition of $Der(\Omega_{[2]}(M))$ as a module of the entire crossed product of $C^\infty(\mathbb{R}^{0|2})$ with $Diff(\mathbb{R}^{0|2})$ we would have to take into account the action of $Mat(2)$ (which is easy) and of R_a 's; the result seems to be more complicated than interesting, so we shall not write it here.

6 Integration and Euler characteristic

This section is devoted to the proof of the following theorem: if γ is a pseudodifferential gorm on a connected manifold M (i.e. a smooth function on $(\Pi T)^2 M$) such that $d_1 \gamma = d_2 \gamma = 0$ (this clearly implies that the restriction of γ to $M \subset (\Pi T)^2 M$, $\gamma|_M$, is a constant), and if moreover γ is integrable and M compact, then

$$\int \gamma = \frac{\gamma|_M}{2} (-\pi)^{m/2} S_m \chi(M), \quad (4)$$

where $\chi(M)$ is the Euler characteristic of M , m is the dimension of M and S_m is the area of the unit m -dimensional sphere.

Let us start with a special case. Let $b_{ij}(x)$ be a Riemann metric on M and let $\beta = b_{ij}(x) d_1 x^i d_2 x^j$; we will prove the theorem for $\gamma = e^{d_1 d_2 \beta}$. If we choose local coordinates so that $b_{ij,k} = 0$ at a given point (Riemann normal coordinates would do), then a simple computation gives that at that point

$$d_1 d_2 \beta = -b_{ij} d_1 d_2 x^i d_1 d_2 x^j - \frac{1}{2} R_{ijkl} d_1 x^i d_1 x^j d_2 x^k d_2 x^l$$

where R_{ijkl} is the curvature of b_{ij} .

If M is compact then $e^{d_1 d_2 \beta}$ is integrable. To compute the integral, first pass to Riemann normal coordinates at a point and integrate over y 's and ξ 's; we end up with the Pfaffian of the curvature, whose integral is well known to be a multiple of the Euler characteristic $\chi(M)$ of M . The result is really

$$\int e^{d_1 d_2 \beta} = \frac{1}{2} (-\pi)^{m/2} S_m \chi(M). \quad (5)$$

It is easy to prove directly that the integral (5) is independent of the choice of the metric. If we add to β an infinitesimal α then $e^{d_1 d_2 (\beta + \alpha)} - e^{d_1 d_2 \beta} = e^{d_1 d_2 \beta} d_1 d_2 \alpha = d_1 (e^{d_1 d_2 \beta} d_2 \alpha)$ and $\int d_1 (e^{d_1 d_2 \beta} d_2 \alpha) = 0$ since $\int u^b \gamma = 0$ for any u and any integrable γ (see section 4.4).

To prove (4) generally, we first have to prove that if δ grows (say) at most polynomially in y 's, $d_1 \delta = d_2 \delta = 0$ and $\delta|_M = 0$, then $\int e^{d_1 d_2 \beta} \delta = 0$. Indeed, since $\delta|_M = 0$, we can find an ϵ such that $\delta = E\epsilon$, where $E = E_1^1 + E_2^2$ is the generator of the scaling $(x^i, \xi_a^i, y^i) \mapsto (x^i, \lambda \xi_a^i, \lambda^2 y^i)$. Since $E = -[d_1, R_1] - [d_2, R_2]$, $\delta = -d_1 R_1 \epsilon - d_2 R_2 \epsilon$ and $\int e^{d_1 d_2 \beta} \delta = \int -d_1 (e^{d_1 d_2 \beta} R_1 \epsilon) - d_2 (e^{d_1 d_2 \beta} R_2 \epsilon) = 0$.

Finally, to prove (4) we just set $\delta = \gamma - \gamma|_M$, multiply β by a constant $s > 0$ and take the limit $s \rightarrow 0_+$.

7 Beyond homological algebra

Differential forms certainly play an important role in topology; they are the basic example of homological algebra, and also of its non-linear generalizations (consider e.g. Maurer-Cartan equation for flat connections, or Sullivan's rational homotopy theory). Since in this paper we "explained" and generalized differential forms, it is also natural to "explain" and generalize homological algebra and its non-linear analogs. As we will see there is a closely related problem: to define differential forms (and gorms and worms) for some generalized manifolds, namely for contravariant functors and more generally for stacks.

It shouldn't be surprising that homological algebra is closely connected with supermanifolds with a right action of the supersemigroup $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ (these objects are really taken from

[Sull]); we shall discuss this connection in section 7.1. The generalization is simply to replace $\mathbb{R}^{0|1}$ with $\mathbb{R}^{0|n}$. The description of differential forms as functions on $M^{\mathbb{R}^{0|1}}$ gives a new point of view on (a part of) homological algebra via representability of functors. It is closely connected with the problem of differential forms (and gorms and worms) on contravariant functors; we shall discuss it in section 7.2.

Let us introduce a part of the picture. By $\mathcal{S}_{[n]}$ we denote the category of supermanifolds with right action of the supersemigroup $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$; morphisms in $\mathcal{S}_{[n]}$ are equivariant maps. We shall view the objects of $\mathcal{S}_{[n]}$ as generalized supermanifolds; this idea is taken directly from Sullivan's rational homotopy theory [Sull] (generalized *manifolds* are then such generalized supermanifolds, on which the parity involution $\mathbb{R}^{0|n} \rightarrow \mathbb{R}^{0|n}$ acts as the parity involution). The idea is as follows. The functor $X \mapsto X^{\mathbb{R}^{0|n}}$ is a fully faithful embedding of the category \mathcal{S} of supermanifolds to the category $\mathcal{S}_{[n]}$. The category $\mathcal{S}_{[n]}$ thus becomes an extension of \mathcal{S} and we can regard objects of $\mathcal{S}_{[n]}$ as generalized supermanifolds; true supermanifolds are the objects of $\mathcal{S}_{[n]}$ of the form $(\Pi T)^n X = X^{\mathbb{R}^{0|n}}$. In other words, we shall treat the objects of $\mathcal{S}_{[n]}$ as if they were of the form $X^{\mathbb{R}^{0|n}}$ for some X . For example, if we have two objects X, Y of $\mathcal{S}_{[n]}$ and two equivariant maps (i.e. two morphisms) between them, a *homotopy* between the maps is an equivariant map $X \times (\Pi T)^n I \rightarrow Y$ that restricts to the two maps at the endpoints of I .

This idea will become more complete and convincing in section 7.2 when we identify objects of $\mathcal{S}_{[n]}$ with the functors $\mathcal{S}^{op} \rightarrow \mathcal{S}$ they represent. In particular, we will get a whole chain of fully faithful embeddings

$$\mathcal{S} \rightarrow \mathcal{S}_{[1]} \rightarrow \mathcal{S}_{[2]} \rightarrow \mathcal{S}_{[3]} \rightarrow \cdots$$

In section 7.4 we will extend it to Lie groupoids and to the stacks they represent.

7.1 Example: $n = 1$ (the case of homological algebra)

Let us consider the case of $n = 1$, corresponding to differential forms, homological algebra etc. Let us start with linear actions of $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$. A right representation of $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$, i.e. a “generalized vector space”, is the same as a non-negatively graded chain complex ($\theta \partial_\theta$ is the degree and ∂_θ is the differential). Two linear equivariant maps $V \rightrightarrows W$ are linearly homotopic, i.e. they are connected by an equivariant map $V \times \Pi T I \rightarrow W$ that is a linear map $V \rightarrow W$ parametrized by $\Pi T I$, iff the two morphisms of chain complexes are homotopic in the usual algebraic sense. If we define homotopy groups of V using equivariant maps $\Pi T S^k \rightarrow V$, they turn out to be the homology groups of the complex V .

Let us now pass to some non-linear actions of $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$. If \mathfrak{g} is a Lie algebra then on $\Pi \mathfrak{g}$ we have a canonical action of $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$, given by identification of $\Pi \mathfrak{g}$ with $(\Pi T G)/G$ (in other words, $C^\infty(\Pi \mathfrak{g}) = \bigwedge \mathfrak{g}^*$, $\theta \partial_\theta$ acts as the degree and ∂_θ as the Chevalley-Eilenberg differential). An equivariant map $\Pi T M \rightarrow \Pi \mathfrak{g}$ is the same as a flat \mathfrak{g} -connection on M : any map $\Pi T M \rightarrow \Pi \mathfrak{g}$ is a \mathfrak{g} -valued differential form on M , and equivariance is easily seen to express the fact that it is a 1-form satisfying the Maurer-Cartan equation. The fundamental group of $\Pi \mathfrak{g}$ is therefore the 1-connected Lie group G , and its higher homotopy groups are the higher homotopy groups of G .

As a little generalization, if $A \rightarrow N$ is a Lie algebroid then again ΠA is an object of $\mathcal{S}_{[1]}$ (this was observed by Vaintrob [Vain]). An equivariant map $\Pi T M \rightarrow \Pi A$ is the same as a Lie algebroid morphism $T M \rightarrow A$; the fundamental groupoid of ΠA is therefore the corresponding Lie groupoid Γ with 1-connected fibres (if it exists).

For more details and examples with interesting higher homotopies, see [Se2].

7.2 Representability of functors and their approximations

For any supermanifolds X, Y we have the supermanifold Y^X of all maps $X \rightarrow Y$ (we shall ignore all problems connected with the fact that Y^X is almost always infinite-dimensional, as they are inessential for our purposes; just imagine that we designed our category \mathcal{S} of supermanifolds so that it contains Y^X for any objects X and Y). Hence any object $Y \in \mathcal{S}$ gives us a functor $\hat{Y} : \mathcal{S}^{op} \rightarrow \mathcal{S}$ given by

$$\hat{Y}(X) = Y^X.$$

Functors of this form are called representable. Y can be reconstructed as $\hat{Y}(point)$, and for example $(IIT)^n Y$ as $\hat{Y}(\mathbb{R}^{0|n})$.

Let \mathcal{SS} denote the category of all functors $F : \mathcal{S}^{op} \rightarrow \mathcal{S}$.² We have described a functor $\mathcal{S} \rightarrow \mathcal{SS}$, $Y \mapsto \hat{Y}$; by Yoneda lemma it is a fully faithful embedding. We can (and will) identify objects of \mathcal{S} with the functors they represent. It is a standard idea to view objects of \mathcal{SS} as generalized objects of \mathcal{S} (a generalized *manifold* is a functor $F : \mathcal{S}^{op} \rightarrow \mathcal{S}$ that preserves the parity involution). In other words, we will understand $F(X)$ as the space of maps from X to some generalized space corresponding to F .³ From this point of view we should define level- n worms on F as functions on $F(\mathbb{R}^{0|n})$. Notice that $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ acts on $F(\mathbb{R}^{0|n})$ from the right, i.e. $F(\mathbb{R}^{0|n})$ is an object of $\mathcal{S}_{[n]}$. In fact the semigroup $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ can be understood as the full subcategory of \mathcal{S} with just one object, $\mathbb{R}^{0|n}$, and $\mathcal{S}_{[n]}$ as the category of contravariant functors from this subcategory to \mathcal{S} ; we restricted F to this subcategory.

Although we understand $F(point)$ as the space of points of F , the functor F is not uniquely specified by $F(point)$ (otherwise all functors would have to be representable), nor is it specified by $F(\mathbb{R}^{0|n})$ for any n . Nevertheless we can use $F(\mathbb{R}^{0|n})$'s to approximate F , since for any X we have the map

$$X^{\mathbb{R}^{0|n}} \times F(X) \rightarrow F(\mathbb{R}^{0|n}),$$

i.e. a map from $F(X)$ to the superspace of $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ -equivariant maps $X^{\mathbb{R}^{0|n}} \rightarrow F(\mathbb{R}^{0|n})$,

$$F(X) \rightarrow Hom_{\mathcal{S}_{[n]}}(X^{\mathbb{R}^{0|n}}, F(\mathbb{R}^{0|n})). \quad (6)$$

Now we can state the definitions. For any object $Y \in \mathcal{S}_{[n]}$, let $\hat{Y} \in \mathcal{SS}$ (the *functor represented by* Y) be the functor given by $\hat{Y}(X) =$ the superspace of all $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ -equivariant maps from $X^{\mathbb{R}^{0|n}}$ to Y ,

$$\hat{Y}(X) = Hom_{\mathcal{S}_{[n]}}(X^{\mathbb{R}^{0|n}}, Y). \quad (7)$$

²We understand these contravariant functors in the strong sense: for any two $X, Y \in \mathcal{S}$ we have a map of supermanifolds $Y^X \times F(Y) \rightarrow F(X)$. Equivalently (using parametrized maps instead of map spaces), for any triple X, Y, Z and any map $Z \times X \rightarrow Y$ we have a map $Z \times F(Y) \rightarrow F(X)$. Morphisms between functors are understood in the strong sense too: a morphism $F_1 \rightarrow F_2$ is a morphism $F_1(X) \rightarrow F_2(X)$ for each X such that for any $Z \times X \rightarrow Y$ the square

$$\begin{array}{ccc} F_1(X) & \rightarrow & F_2(X) \\ \uparrow & & \uparrow \\ Z \times F_1(Y) & \rightarrow & Z \times F_2(Y) \end{array}$$

commutes.

³ $F(X)$ can be seen as just an approximation to the generalized space of maps F^X defined by $F^X(Y) = F(X \times Y)$, i.e. $F(X) = F^X(point)$

(the functor $\mathcal{S}_{[n]} \rightarrow \mathcal{SS}$, $Y \mapsto \hat{Y}$, is the right adjoint of the restriction functor $\mathcal{SS} \rightarrow \mathcal{S}_{[n]}$). Functors of the form \hat{Y} will be called *representable at level n* . Notice that they can be expressed in terms of worms of level n (if we choose coordinates on Y , a map $X^{\mathbb{R}^{0|n}} \rightarrow Y$ becomes a collection of functions on $X^{\mathbb{R}^{0|n}}$, i.e. a collection of level- n worms on X). Representability at level 0 is, of course, the ordinary representability. If $F : \mathcal{S}^{op} \rightarrow \mathcal{S}$ is representable at level n , the corresponding object $Y \in \mathcal{S}_{[n]}$ can be found as $F(\mathbb{R}^{0|n})$. For any functor $F : \mathcal{S}^{op} \rightarrow \mathcal{S}$, the functor $F_{[n]} = \widehat{F(\mathbb{R}^{0|n})}$ will be called *the n -th approximation of F* . The equation (6) gives us a natural morphism $F \rightarrow F_{[n]}$; F is representable at level n iff the morphism is an isomorphism.

The morphism $F \rightarrow F_{[n]}$ becomes an isomorphism when we restrict F and $F_{[n]}$ to the full subcategory $\mathcal{D}_n \subset \mathcal{S}$ of supermanifolds of dimension at most $0|n$ (we have to show that $F(\mathbb{R}^{0|k}) \rightarrow F_{[n]}(\mathbb{R}^{0|k})$ is a diffeomorphism whenever $k \leq n$; for $k = n$ it is tautological, and for other k 's it is enough to choose maps $\mathbb{R}^{0|k} \rightarrow \mathbb{R}^{0|n} \rightarrow \mathbb{R}^{0|k}$ that compose to identity on $\mathbb{R}^{0|k}$). As a consequence, representability at level n implies representability at all higher levels. By taking successive approximations we get a chain of morphisms

$$\cdots \rightarrow F_{[3]} \rightarrow F_{[2]} \rightarrow F_{[1]} \rightarrow F_{[0]}; \quad (8)$$

together with the morphisms $F \rightarrow F_{[n]}$ it forms a commutative diagram.

As a final remark, we could repeat these definitions from a more natural point of view. The idea is to approximate functors $\mathcal{S}^{op} \rightarrow \mathcal{S}$ (objects of \mathcal{SS}) by their restrictions to the full subcategories \mathcal{D}_n of \mathcal{S} defined above that form a chain of inclusions

$$\mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots \subset \mathcal{S}.$$

The category $\mathcal{D}_n\mathcal{S}$ of all functors $\mathcal{D}_n^{op} \rightarrow \mathcal{S}$ is equivalent to $\mathcal{S}_{[n]}$ (the equivalence $\mathcal{D}_n\mathcal{S} \rightarrow \mathcal{S}_{[n]}$ is given by restriction), so we would get equivalent definitions.

7.3 Examples of approximations

Let G be a Lie group and let $F : \mathcal{S}^{op} \rightarrow \mathcal{S}$ be given by $F(X) = (G^X)/G$. To compute the n -th approximation of F we just have to compute the space $F(\mathbb{R}^{0|n})$ and the right action of $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ on this space. Since $F(\text{point}) = \text{point}$, $F_{[0]}(X) = \text{point}$. The first approximation is more interesting: $F(\mathbb{R}^{0|1}) = (\Pi TG)/G = \Pi \mathfrak{g}$, therefore $F_{[1]}(X)$ is the space of flat \mathfrak{g} -connections on X . Since locally (in X) one cannot distinguish F from $F_{[1]}$, all higher approximations of F are just $F_{[1]}$.

As a small generalization, let Γ be a Lie groupoid. For any X let $X \times X$ be the pair groupoid (with X as the space of objects and with one arrow between any two objects) and finally let $F(X) = \text{Hom}(X \times X, \Gamma)$. Then $F(\text{point})$ is the base of Γ (the space of its objects), hence $F_{[0]}$ is just the functor represented by the base. To compute $F_{[1]}$ notice that $F(\mathbb{R}^{0|1}) = \Pi A$ where A is the Lie algebroid corresponding to Γ . Thus we found that $F_{[1]}(X)$ is the space of all Lie algebroid morphisms $TX \rightarrow A$. Since locally we cannot distinguish between these Lie algebroid morphisms and Lie groupoid morphisms $X \times X \rightarrow \Gamma$, all higher approximations of F are again equal to $F_{[1]}$.

The next example is trivially representable at level 1, but it is interesting for other reasons. Let G be a Lie group, \mathfrak{g} its Lie algebra, and let $F(X)$ be the space of \mathfrak{g} -connections on X (i.e. the space of \mathfrak{g} -valued 1-forms on X). On $F(X)$ we have action of the group G^X (by gauge transformations); if we understand F as a generalized space, it means that G acts on

F . The algebra of differential forms on F , i.e. of functions on $F(\mathbb{R}^{0|1})$, is the Weil algebra $W(\mathfrak{g})$; the actions of $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ and of $\Pi TG = G^{\mathbb{R}^{0|1}}$ give rise to its standard G -differential algebra structure. If M is a manifold with an action of G , we can consider the generalized space $F \times M$ (given by $(F \times M)(X) = F(X) \times M^X$); since G acts on both F and M , it acts also on $F \times M$. The complex of basic forms on $F \times M$, i.e. of ΠTG -invariant functions on $(F \times M)(\mathbb{R}^{0|1}) = F(\mathbb{R}^{0|1}) \times \Pi TM$, is the basic subcomplex in the Weil model of equivariant cohomology (notice that F behaves as if it were EG). To get Cartan model, notice that any connection on $\mathbb{R}^{0|1}$ can be made to vanish at the origin, using a suitable gauge transformation; the space of such connections can be identified with \mathfrak{g} (any such connection is of the form $t\theta d\theta$, where $t \in \mathfrak{g}$). After we impose this condition, only the group of constant gauge transformations $G \subset \Pi TG$ remains. ΠTG -invariant functions on $F(\mathbb{R}^{0|1}) \times \Pi TM$ can thus be identified with G -invariant functions on $\mathfrak{g} \times \Pi TM$; the latter is the Cartan model.

Let us pass to some examples where $F_{[2]}$ is different from $F_{[1]}$. Let $F(X) = \Gamma(S^2 T^* X)$ (or $\Gamma(T_{\square\square}^* X)$ in the notation of section 5). Then $F(\text{point}) = F(\mathbb{R}^{0|1}) = 0$. On the other hand, as we have found in section 5, $F = F_{[2]}$ since $F_{[2]}(X) = \Gamma(\widehat{T_{\square\square}^* X})$ and $\widehat{T_{\square\square}^*} = T_{\square\square}^*$. The functor F is thus representable at level 2. As another example, let $F(X) = \Gamma(T_{\boxplus}^* X)$. Then again $F(\text{point}) = F(\mathbb{R}^{0|1}) = 0$, but $F_{[2]}(X) = \Gamma(\widehat{T_{\boxplus}^* X})$. This time F is different from its second approximation; one can prove that it is representable at level 3.

Let us finish with a simple example of a functor F for which the chain of morphisms (8) doesn't stabilize. Let $F(X) = C^\infty(X \times X)$. Then $F_{[k]}(X) = \Gamma(J^k(X))$, where $J^k(X) \rightarrow X$ is the vector bundle of k -jets of functions on X (i.e. $F_{[k]}(X)$ is the space of functions on the k th formal neighbourhood of the diagonal in $X \times X$.)

7.4 Approximations and representability of stacks

This section can be seen in two ways. Above we defined the categories $\mathcal{S}_{[n]}$ as generalizations of the category \mathcal{S} of supermanifolds. Here we extend these generalizations from supermanifolds to Lie supergroupoids (any (super)manifold can be seen as a Lie (super)groupoid, with identity arrows only). We define 2-categories $\mathcal{G}_{[n]}$ as generalizations of the 2-category \mathcal{G} of Lie supergroupoids. The objects of $\mathcal{G}_{[n]}$ will be Lie supergroupoids over the supersemigroup $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ (roughly speaking, Lie supergroupoids on which $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ acts up to natural transformations). In the case of $n = 1$ they will include some interesting known examples where “ $d^2 = 0$ up to gauge transformations”, e.g. Cartan model of equivariant cohomology, or quasi-Poisson groupoids.

The other point of view (extending the section 7.2) is to look at the stacks represented by the objects of $\mathcal{G}_{[n]}$. Principal G -bundles (for some fixed Lie group G) are perhaps the simplest example of a stack, and this stack is representable in the appropriate way: a principal bundle $P \rightarrow M$ is the same as a 1-morphism of groupoids $M \rightarrow G$ in the sense of Hilsen and Skandalis (see below). We can get other interesting stacks by considering 1-morphisms $M \rightarrow \Gamma$ for a Lie groupoid Γ .

The stack of principal G -bundles with a choice of a connection is, however, not representable in this sense. Fortunately, it is representable at level 1, i.e. by an object of $\mathcal{G}_{[1]}$. This object can be found in a tautological way (just as in 7.2, where we would find an object of $\mathcal{S}_{[1]}$, corresponding to a functor F , as $F(\mathbb{R}^{0|1})$), as the supergroupoid of all principal G -bundles over $\mathbb{R}^{0|1}$ with a choice of connection. Any principal G -bundle over $\mathbb{R}^{0|1}$ is trivializable so we

can consider just connections on the trivial bundle (and get equivalent groupoid). The objects of this groupoid are thus \mathfrak{g} -valued 1-forms on $\mathbb{R}^{0|1}$, the arrows are given by gauge transformations (i.e. by the action of the supergroup $\Pi TG = G^{\mathbb{R}^{0|1}}$); the supersemigroup $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ acts from the right on this supergroupoid, so we get an object of $\mathcal{G}_{[1]}$. We can get a yet smaller equivalent groupoid by considering only those \mathfrak{g} -valued 1-forms that vanish at the origin (see the example in the section 7.3); this is still an object of $\mathcal{G}_{[1]}$, as $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ acts on it up to gauge transformations.

Finally we have to say that this section is not logically complete (roughly because we do not give a definition of a “superstack”, so it is not clear, how to restrict a stack to $\mathbb{R}^{0|n}$ to get an object of $\mathcal{G}_{[n]}$ (even though it is clear in examples); on the other hand, the definition of a stack represented by an object of $\mathcal{G}_{[n]}$ is all right). We hope that this fault can be excused by interesting examples.

7.4.1 Groupoids over a category and Hilsum-Skandalis morphisms

Let us recall from [Gro] that a functor $F : \mathcal{E} \rightarrow \mathcal{F}$ is a *fibration of \mathcal{F} by groupoids*, (or shortly, a *groupoid over \mathcal{F}*), if it satisfies the following lifting property for morphisms: for any morphism $f : X \rightarrow Y$ in \mathcal{F} and any object Q in \mathcal{E} such that $F(Q) = Y$ there is a morphism $\tilde{f} : P \rightarrow Q$ in \mathcal{E} such that $F(\tilde{f}) = f$, and moreover \tilde{f} is essentially unique, i.e. if $\tilde{f}' : P' \rightarrow Q$ is another morphism such that $F(\tilde{f}') = f$ then there is unique $h : P' \rightarrow P$ such that $\tilde{f} \circ h = \tilde{f}'$ and $F(h) = id_X$. As a simple example, we can take \mathcal{E} to be the category of principal G -bundles (with equivariant maps as morphisms) and \mathcal{F} the category of manifolds; the functor $F : \mathcal{E} \rightarrow \mathcal{F}$ assigns to a principal bundle its base.

For any object X of \mathcal{F} let $F^{-1}(X)$ be the fibre above X , i.e. the subcategory of \mathcal{E} of objects P such that $F(P) = X$ and morphisms f such that $F(f) = id_X$. It is easy to see that all $F^{-1}(X)$ ’s are groupoids. If $f : X \rightarrow Y$ is a morphism then by lifting f at all objects of $F^{-1}(Y)$ we get a functor $F^{-1}(Y) \rightarrow F^{-1}(X)$; if we choose the lifts differently, we get an isomorphic functor. By lifting all the morphisms of \mathcal{F} we get a lax functor from \mathcal{F}^{op} to the category of groupoids; in the opposite direction, any (lax) functor from \mathcal{F}^{op} to the category of groupoids gives us a groupoid over \mathcal{F} .

If Δ_1 is the “segment” category with objects 0 and 1 and with only one non-identity morphism, $0 \rightarrow 1$, then a groupoid over Δ_1 is called a *Hilsum-Skandalis (HS) morphism* from $F^{-1}(1)$ to $F^{-1}(0)$; it is essentially a functor $F^{-1}(1) \rightarrow F^{-1}(0)$, but the functor depends (up to natural transformations) on the choice of lifts.

Groupoids form a (weak) 2-category, with HS morphisms as 1-morphisms (if $\mathcal{C} \rightarrow \Delta_1$ and $\mathcal{D} \rightarrow \Delta_1$ are two HS morphisms from Γ_1 to Γ_2 , a 2-morphism between them is an isomorphism $\mathcal{C} \cong \mathcal{D}$ that is identity on both Γ_1 and Γ_2). If Δ_2 is the “triangle” category, with three objects 0, 1 and 2, and with three morphisms (except for identities)

$$\begin{array}{ccc} 0 & \rightarrow & 1 \\ & \searrow & \downarrow \\ & & 2 \end{array},$$

a groupoid over Δ_2 gives us 3 HS morphisms, $F^{-1}(1) \rightarrow F^{-1}(0)$, $F^{-1}(2) \rightarrow F^{-1}(1)$ and $F^{-1}(2) \rightarrow F^{-1}(0)$. The HS morphism $F^{-1}(2) \rightarrow F^{-1}(0)$ is then a *composition* of $F^{-1}(2) \rightarrow F^{-1}(1)$ and $F^{-1}(1) \rightarrow F^{-1}(0)$. It is easy to see that a composition of two HS morphisms $\Gamma_2 \rightarrow \Gamma_1 \rightarrow \Gamma_0$ always exists and it is unique up to a canonical isomorphism: we have to

define the arrows over $0 \rightarrow 2$ (i.e. arrows $X \rightarrow Z$, where X is an object of Γ_0 and Z of Γ_2); these will be, by definition, pairs of arrows $X \rightarrow Y \rightarrow Z$, where we identify $X \rightarrow Y_1 \rightarrow Z$ with $X \rightarrow Y_2 \rightarrow Z$ whenever there is an arrow $Y_1 \rightarrow Y_2$ such that the two triangles

$$\begin{array}{ccccc} X & \rightarrow & Y_1 & & \\ & \searrow & \downarrow & \swarrow & \\ & & Y_2 & \rightarrow & Z \end{array}$$

are commutative.

7.4.2 The 2-category of Lie groupoids and the stacks they represent

A *Lie groupoid over a category \mathcal{F}* is a groupoid over \mathcal{F} , $\mathcal{E} \rightarrow \mathcal{F}$, such that for any arrow $f : X \rightarrow Y$ of \mathcal{F} the set of arrows of \mathcal{E} over f is a manifold, and the composition of arrows in \mathcal{E} is smooth (i.e. the fibres of $\mathcal{E} \rightarrow \mathcal{F}$ are Lie groupoids and they act smoothly on the manifolds of arrows, for which the composition is defined). A Hilsum-Skandalis morphism of Lie groupoids is then simply a Lie groupoid over Δ_1 . For example, a HS morphism $M \rightarrow G$, where M is a manifold (understood as a Lie groupoid with identity arrows only) and G a Lie group, is the same as a principal G -bundle over M (the bundle is the manifold of arrows over $0 \rightarrow 1$). Lie supergroupoids with HS morphisms form a (weak) 2-category, denoted \mathcal{G} .

As a generalization of principal G -bundles, any Lie groupoid Γ defines a stack (a groupoid over the category of manifolds, satisfying a sheaf-like condition): The objects of this category are HS morphisms $M \rightarrow \Gamma$ (where M runs over all manifolds) and morphisms are commutative triangles

$$\begin{array}{ccc} M_1 & \rightarrow & \Gamma \\ \downarrow & \nearrow & \\ M_2 & & \end{array}.$$

Stacks of this form will be called *representable at level 0*. If M is contractible, the groupoid of all HS morphisms $M \rightarrow \Gamma$ (i.e. the fibre of the stack over M) is equivalent to the groupoid of all (ordinary) maps $M \rightarrow \Gamma$. Notice that Γ is equivalent to the fibre over $M = \text{point}$. For a review of the 2-category \mathcal{G} and of the stacks represented by Lie groupoids, see [Met].

Notice that to describe a HS morphism $M \rightarrow \Gamma$ we need to give the space P of arrows over $0 \rightarrow 1$, the submersion $P \rightarrow M$ (the map sending arrows to their heads) and the action of Γ on P (composition of arrows); we get a HS morphism iff the submersion $P \rightarrow M$ is surjective and Γ acts freely and transitively on each of its fibres. A space P with these properties is (for obvious reasons) often called a *principal Γ bundle*.

As an example, if a group G acts on a space V , we can consider the stack whose objects are principal G -bundles with equivariant maps to V ; this stack can be represented by the action groupoid of G on V .

7.4.3 Generalized Lie groupoids and the stacks they represent

We shall need Lie groupoids over some smooth categories. Rather than defining general Lie categories (where, for our purposes, dimension should be allowed to be different on different components), we do it just in special cases that we really need, where the objects of the base category \mathcal{F} form a discrete set. So suppose \mathcal{F} is a Lie category of this kind, i.e. simply a

category enriched in the category of manifolds.⁴ A *Lie groupoid over \mathcal{F}* is a groupoid over \mathcal{F} , $F : \mathcal{E} \rightarrow \mathcal{F}$, such that the set of arrows $F^{-1}(\text{Hom}(X, Y))$ is a manifold (for any objects X, Y of \mathcal{F}), the projection $F^{-1}(\text{Hom}(X, Y)) \rightarrow \text{Hom}(X, Y)$ is a submersion and the composition of arrows in \mathcal{E} is smooth.

Now we can define the 2-categories $\mathcal{G}_{[n]}$ of generalized Lie (super)groupoids. The objects of $\mathcal{G}_{[n]}$ are the Lie supergroupoids over the supersemigroup $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ (understood as a category with just one object; as we have seen above, it is useful to identify it with the full subcategory of \mathcal{S} with the object $\mathbb{R}^{0|n}$). Morphisms (generalized HS morphisms) are Lie supergroupoids over $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}} \times \Delta_1$; their compositions are defined as Lie supergroupoids over $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}} \times \Delta_2$. The *fibre* \mathbf{G}_0 of an object \mathbf{G} of $\mathcal{G}_{[n]}$ is the fibre of $\mathbf{G} \rightarrow (\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ over the unique object of $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$.

If \mathbf{G} is a Lie supergroupoid on which $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ acts, it gives us an object $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}} \ltimes \mathbf{G} \rightarrow (\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ of $\mathcal{G}_{[n]}$ (after all, objects of $\mathcal{G}_{[n]}$ are supergroupoids on which $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ acts up to natural transformations); we will denote it for short $\underline{\mathbf{G}}$ (notice that the fibre of $\underline{\mathbf{G}}$ is \mathbf{G}). We get in this way an embedding $\mathcal{G} \rightarrow \mathcal{G}_{[n]}$: for any groupoid Γ , the corresponding object of $\mathcal{G}_{[n]}$ is $\underline{\Gamma}^{\mathbb{R}^{0|n}}$. Any object \mathbf{G} of $\mathcal{G}_{[n]}$ now defines a stack of all 1-morphisms $\underline{M}^{\mathbb{R}^{0|n}} \rightarrow \mathbf{G}$; these stacks will be called *representable at the level n* .

We conclude with a concrete description of 1-morphisms $\underline{M}^{\mathbb{R}^{0|n}} \rightarrow \mathbf{G}$. By definition, we have to describe the superspace of arrows over $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}} \times (0 \rightarrow 1)$ and their composition with arrows in $\underline{M}^{\mathbb{R}^{0|n}}$ and in \mathbf{G} . However, we can naturally identify this superspace of arrows with $P \times (\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$, where P is the superspace of arrows over $id \times (0 \rightarrow 1)$.

A morphism $\underline{M}^{\mathbb{R}^{0|n}} \rightarrow \mathbf{G}$ is thus equivalently given by a surjective submersion $P \rightarrow M^{\mathbb{R}^{0|n}}$ and a right action of \mathbf{G} on P satisfying two conditions: 1. the map $P \rightarrow M^{\mathbb{R}^{0|n}}$ is \mathbf{G} -equivariant (where \mathbf{G} acts on $M^{\mathbb{R}^{0|n}}$ via the projection $\mathbf{G} \rightarrow (\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ and via the action of $(\mathbb{R}^{0|n})^{\mathbb{R}^{0|n}}$ on $M^{\mathbb{R}^{0|n}}$) and 2. the action of the fibre \mathbf{G}_0 of \mathbf{G} on P makes $P \rightarrow M^{\mathbb{R}^{0|n}}$ to a principal \mathbf{G}_0 -bundle.

7.5 Examples of stacks and of generalized Lie groupoids

7.5.1 “Categorified de Rham complex”

This is one of the simplest possible examples. Let $\mathbb{R}[k] \subset \Omega^k(\mathbb{R}^{0|1})$ denote the (1-dimensional) group of closed k -forms on $\mathbb{R}^{0|1}$. Since $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ acts from the right on $\mathbb{R}[k]$, it is an object of $\mathcal{S}_{[1]}$, but since $\mathbb{R}[k]$ is a group, it gives us an object of $\mathcal{G}_{[1]}$, namely $\underline{\mathbb{R}[k]}$. An $\mathcal{S}_{[1]}$ -morphism $\Pi T M \rightarrow \underline{\mathbb{R}[k]}$ is, tautologically, a closed k -form on M . Here we shall consider 1-morphisms $\underline{\Pi T M} \rightarrow \underline{\mathbb{R}[k]}$ in the 2-category $\mathcal{G}_{[1]}$. As we explained above, to describe such a morphism, we just have to describe the supermanifold P of arrows over $id \times (0 \rightarrow 1)$. The result is a principal $\mathbb{R}[k]$ -bundle $P \rightarrow \Pi T M$ in the category $\mathcal{S}_{[1]}$, i.e. $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ acts from the right on P and the maps $P \rightarrow \Pi T M$ and $\mathbb{R}[k] \times P \rightarrow P$ are $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ -equivariant. Such P 's are easily seen to be classified by $H^{k+1}(M, \mathbb{R})$ (see [Se2]).

⁴this means that $\text{Hom}(X, Y)$ is a manifold for any objects X and Y of \mathcal{F} and that $\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$ is a smooth map

Let us consider the exact sequence

$$0 \rightarrow \mathbb{R}[k] \rightarrow \Omega^k(\mathbb{R}^{0|1}) \rightarrow \mathbb{R}[k+1] \rightarrow 0.$$

All 1-morphisms $\Pi TM \rightarrow \Omega^k(\mathbb{R}^{0|1})$ are easily seen to be isomorphic. A lift of a 1-morphism $\Pi TM \rightarrow \Omega^k(\mathbb{R}^{0|1})$ to $\Pi TM \rightarrow \mathbb{R}[k]$ is equivalent to trivialization of the composed morphism $\Pi TM \rightarrow \Omega^k(\mathbb{R}^{0|1}) \rightarrow \mathbb{R}[k+1]$, i.e. to an isomorphism (a 2-morphism) between the composition $\Pi TM \rightarrow \mathbb{R}[k+1]$ and the trivial morphism $\Pi TM \rightarrow \mathbb{R}[k+1]$ (which is the same as a trivialization of the corresponding principal $\mathbb{R}[k+1]$ -bundle over ΠTM).

7.5.2 Weil and Cartan models of equivariant cohomology

We met this example in section 7.3, here we just rephrase it from a different (and more natural) point of view. Let a Lie group G act on a manifold N ; let us consider the stack of principal G -bundles with equivariant maps to N , in other words, the stack of HS morphisms $M \rightarrow \Gamma$, where Γ is the action groupoid Γ of G on N .

Objects of this stack have a lot of automorphisms. Let us consider another, “rigidified” stack of principal G -bundles with equivariant maps to N (as above) and with a choice of connection. This stack is more rigid (any automorphism is specified by its restriction to a point, if the base is connected), but it is in the obvious sense homotopy equivalent to the first one.

This rigidified stack is representable at level 1. To find the corresponding object of $\mathcal{G}_{[1]}$, we have to take the groupoid whose objects are triples (P, h, α) , where $P \rightarrow \mathbb{R}^{0|1}$ is a principal G -bundle, $h : P \rightarrow N$ is an equivariant map and α a connection on P . To get a small equivalent groupoid, take P to be the trivial bundle (any bundle over $\mathbb{R}^{0|1}$ can be trivialized): now the objects are pairs (h, α) , h is a map $\mathbb{R}^{0|1} \rightarrow N$ and α a \mathfrak{g} -valued 1-form on $\mathbb{R}^{0|1}$, i.e. they form the supermanifold $\Pi TN \times \Pi T\Pi\mathfrak{g}$ (functions there are differential forms on N with values in the Weil algebra of \mathfrak{g}). The invariant functions on $\Pi TN \times \Pi T\Pi\mathfrak{g}$ are the basic subcomplex, i.e. we got Weil model of equivariant cohomology. Notice that we can get even smaller equivalent groupoid: we can always make α to vanish at the origin by a gauge transformation. By taking only these α ’s, we get Cartan model.

Equivariant cohomology of N is the cohomology of the classifying space of the action groupoid of G on N . It seems probable that one can get cohomology of classifying spaces of other (proper) Lie groupoids by similar means, i.e. by replacing the classifying stack of HS morphisms $M \rightarrow \Gamma$ with a homotopy equivalent rigid stack (whatever is the precise definition). Unfortunately, we were not able to solve this problem, so we leave it for later work.

7.5.3 Quasi-Poisson groupoids

As was noticed by A. Vaintrob [Vain], a Lie algebroid structure on a vector bundle $A \rightarrow M$ is equivalent to an odd Poisson structure on ΠA^* of degree 1. In this section we shall use usual notation for \mathbb{Z} -graded supermanifolds, i.e. we shall denote ΠA^* as $A^*[1]$ (a \mathbb{Z} -grading can be seen as an action of the “scaling” 1-parameter group; degrees are then weights with respect to this action). A Lie quasi-bialgebroid structure on A (as was observed in [Se1]) is equivalent to a principal $\mathbb{R}[2]$ -bundle $X \rightarrow A^*[2]$ in the category of graded supermanifolds, with an $\mathbb{R}[2]$ -invariant odd Poisson structure on X ; this Poisson structure can be projected to $A^*[1]$, making A to a Lie algebroid.

Lie quasi-bialgebroids are understood as infinitesimal objects corresponding to quasi-Poisson groupoids. We can give a simple invariant definition of the latter by integrating X to an odd symplectic groupoid, keeping track of the grading and of the $\mathbb{R}[2]$: a *quasi-Poisson structure* on a Lie groupoid $\Gamma \rightrightarrows M$ is a graded odd symplectic groupoid $\check{\Gamma}$ with symplectic form of degree 1, with a submersion $\check{\Gamma} \rightarrow \mathbb{R}[-1]$ that is a morphism of groupoids ($\check{\Gamma} \rightarrow \mathbb{R}[-1]$ is the moment map for an action of $\mathbb{R}[2]$ on $\check{\Gamma}$) and with an isomorphism $\check{\Gamma}/\!\!/ \mathbb{R}[2] \cong T^*[1]\Gamma$.

Quasi-Poisson groupoids give us simple examples of objects of $\mathcal{G}_{[1]}$. We only need a part of the structure of a quasi-Poisson groupoid: a graded Lie supergroupoid $\check{\Gamma}$ over $\mathbb{R}[-1]$ such that the action of the scaling group on the fibre of $\check{\Gamma}$ can be extended to action of the scaling semigroup (\mathbb{R}, \times) . We get a groupoid over $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ using the isomorphism $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}} = (\mathbb{R}, \times) \ltimes \mathbb{R}[-1]$ (a general element of $(\mathbb{R}^{0|1})^{\mathbb{R}^{0|1}}$ is of the form $\theta \mapsto a\theta + \beta$; (\mathbb{R}, \times) corresponds to $\theta \mapsto a\theta$ and $\mathbb{R}[-1]$ to $\theta \mapsto \theta + \beta$).

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